DISCONTINUOUS FRACTAL FUNCTIONS AND FRACTAL HISTOPOLATION

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Abstract. Fractal functions that produce smooth and non-smooth approximants constitute an advancement to classical nonrecursive methods of approximation. In both classical and fractal approximation methods emphasis is given for investigation of continuous approximants whereas much real data demand discontinuous models. This article intends to point out that many of the results on fractal functions in the traditional setting can be immediately extended to the discontinuous case. Another topic is the study of area matching properties of integrable fractal functions in order to introduce the concept of fractal histopolation.

1. Introductory Remarks

Approximation theory, which primarily focuses on the approximation of real-valued continuous functions by some simpler class of functions, covers a great deal of mathematical territory. Beyond polynomials, trigonometric functions and splines, multitudes of approximation tools were developed which were often directed to a specific application domain. In the historical development of "classical" approximation theory, all efforts were directed towards study of smooth approximants. On the other hand, many experimental and real world signals rarely show sensation of smoothness in their traces and hence demand rough functions for an effective representation.

Following Benoit Mandelbrot's fractal vision of the universe, the notion of fractal function was introduced in reference [1]. Subsequently, it was observed that fractal functions can be used for smooth approximation as well [6], thereby supplementing various traditional approximation techniques. For more than a quarter century of its first pronouncement, theory of fractal functions has been extensively researched and has evolved beyond its mathematical framework; see, for instance, [5, 9, 10, 13, 16, 18, 25, 26]. Specific applications of fractal interpolation function include medicine, physics, and economics with, for instance, in the study of tumor perfusion [11], electroencephalograms [23], turbulence [8], speech signals [24], signal processing [21], stock market [27] etc.

The fractal functions were originally introduced as continuous functions interpolating a prescribed set of data. Thus, the approximation methods, both fractal functions and their precedents, are based on continuity, and methods for discontinuous approximation are limited. However, many real data requires to be modeled by discontinuous functions. For instance, discontinuous functions arise as solutions of partial differential equations describing different types of systems from classical physics. Since machines have only finite precision, any functions as represented by machine will be discontinuous and so is anything which involves discretization. Mandelbrot's view of the price movement in competitive markets is one of the profound sources of discontinuity. In words of Mandelbrot [15]: "...But prices on competitive markets need not be continuous, and they are conspicuously discontinuous. The only reason for assuming continuity is that many sciences tend, knowingly or not, to copy the procedures that prove successful in Newtonian physics...." Hence it appears that Mandelbrot envisaged not only a world of irregularity, but ultimate discontinuity.

Deriving principle influence from these facts, the current article intends to investigate discontinuous fractal functions and their elementary properties. These fractal functions may not interpolate the data set, thus breaking the inherent continuity and interpolatory nature of the fractal functions in the traditional setting. Discontinuous fractal functions reported recently in [19] provides one of the impetus for the current study. However, our main aim is to show that the tools developed in 1986 paper [1] are well-suited to deal with discontinuous fractal functions as well. Further, we convey that the algebraic structure of equations governing fractal functions

and their moment integrals are similar to that of usual continuous fractal interpolation function, but provides extra degrees of freedom. To this end, first we prove that the set of points of discontinuity of these fractal functions has Lebesgue measure zero; a fact that ensures Riemann integrability. As a consequence, functional equations for various integral transforms of these discontinuous fractal functions can be easily derived, thus finding potential applications in various fields in science and engineering. For a restricted class of discontinuous fractal functions and its connection with Weierstrass type functional equations, the reader is invited to refer [3].

Interpolation "represents" a function by preserving function values at prescribed knot points. A closely related but different approximation method is Histopolation which preserves integrals of the function over the intervals of histopolation. Given a mesh $\Delta := \{x_0, x_1, \dots, x_N\}$ with strictly increasing knots and a histogram $F = \{f_1, f_2, \dots, f_N\}$, that is f_i is the frequency for the interval $[x_{i-1}, x_i]$ with mesh spacing h_i, $i = 1, 2, \dots, N$, histopolation seeks to find a function f that satisfies the "area" matching condition:

$$\int_{x_{i-1}}^{x_i} f(x)\ \mathrm{d}x = f_i h_i.$$

There are several models that lead to histopolation problem. For instance, F may be obtained from a finite sample with the observed frequency f_i in the class interval $[x_{i-1}, x_i)$ for $i = 1, 2, \dots, N$ and area matching function f may be taken as approximation of the unknown density function of the underlying random variable. The following fundamental problem occurring in one dimensional motion of a material point is another example where the problem of histopolation emerges quite naturally. Suppose we have to model velocity $f(x)$ of the material point wherein $g(x)$ is the position of the point at time x, which is known at specified points $x_0 < x_1 < \dots < x_N$. Then we have

$$g(x) = g(x_0) + \int_{x_0}^{x} f(t)\ \mathrm{d}t, \quad x_0 \le t \le x.$$

Since

$$\int_{x_{i-1}}^{x_i} f(x)\ \mathrm{d}x = g(x_i) - g(x_{i-1}), \quad i = 1, 2, \dots, N$$

a representative for f can be obtained by solving the histopolation problem with values in the histogram F given by

$$f_i = \frac{g(x_i) - g(x_{i-1})}{h_i}.$$

In contrast to the vast literature available on interpolation, the researches on histopolation is limited. In case of smooth histopolant f, the problem of histopolation can be easily transformed in to a problem of interpolation as follows. If we construct an interpolant $g \in \mathcal{C}^1[x_0, x_N]$ with interpolation conditions $g(x_i) = \sum_{j=1}^{i} h_j f_j + g(x_0)$, $i = 1, 2, \dots, N$, where $g(x_0)$ is arbitrary, then $f = g'$ solves the histopolation problem. This may be one of the reasons for the obscurity of histopolation as a separate problem. However, the situation is different in the case of fractal functions as they are not differentiable in general. At the same time, as in the case of interpolation, constructing rough histopolants is of practical relevance. Owing to these reasons, fractal histopolation deserves special attention and it is to this that the last section of the paper focus on.

Overall, the current study may be viewed as an attempt to revitalize fractal functions and their applications and to initiate a study on fractal histopolation.

2. Continuous Fractal Interpolation Function: Revisited

To make the article fairly self-contained, we shall briefly evoke the notion of Fractal Interpolation Function and associated concepts in this section. To prepare the setting, first we need the following definition.

Definition 2.1. Let (X, d) be a complete metric space. Let $m > 1$ be a positive integer and let $w_i : X \to X$ for $i = 1, 2, \dots, m$ be continuous mappings. Then the collection $\{X; w_1, w_2, \dots, w_m\}$ is called an Iterated Function System, IFS for short.

For a given IFS $\mathcal{F} = \{X; w_1, w_2, \dots, w_m\}$ one can associate a set-valued map, which is termed collage map, as follows. Let $\mathcal{H}(X)$ denote the collection of all non-empty compact subsets of X endowed with the Hausdorff metric

$$h(A,B) = \max\Big\{\max_{a\in A}\min_{b\in B} d(a,b), \max_{b\in B}\min_{a\in A} d(b,a)\Big\}\ \forall A, B \in \mathcal{H}(X).$$

Define $W : \mathcal{H}(X) \to \mathcal{H}(X)$ by

$$W(B) = \cup_{i=1}^m w_i(B),$$

where $w_i(B) = \{w_i(b) : b \in B\}$.

Definition 2.2. A nonempty compact subset A of X is called an attractor of an IFS $\mathcal{F} = \{X; w_1, w_2, \dots, w_m\}$ if

(1) A is a fixed point of W, that is $W(A) = A$
(2) there exists an open set $U \subseteq X$ such that $A \subset U$ and $\lim_{k\to\infty} h\big(W^k(B), A\big) = 0$ for all $B \in \mathcal{H}(U)$, where W^k is the k-fold autocomposition of W.

The largest open set U for which (2) holds is called the basin of attraction for the attractor A of the IFS $\mathcal{F}$. The attractor A is also referred to as a fractal or self-referential set owing to the fact that A is a union of transformed copies of itself.

If the IFS $\mathcal{F}$ is contractive (hyperbolic), that is each map w_i is a contraction map, then the existence of a unique attractor is ensured by the Banach fixed point theorem and in this case the basin of attraction is X.
In what follows, the question of how to obtain continuous functions whose graphs are fractals in the above sense is readdressed.
Let $\{(x_i, y_i) : i = 0, 1, \dots, N\}$ denote the cartesian coordinates of a finite set of points with increasing abscissae in the Euclidean plane $\mathbb{R}^2$. Let I denote the closed bounded interval $[x_0, x_N]$ and $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, \dots, N$. Suppose $L_i : I \to I_i$ be contraction homeomorphism such that

$$\begin{cases} L_i(x_0) = x_{i-1}, \quad L_i(x_N) = x_i, \\ |L_i(x) - L_i(x^*)| \le l_i|x - x^*| \quad \forall\, x, x^* \in I,\ l_i \in (0,1). \end{cases} \tag{2.1}$$

Note that $\{I; L_i, i = 1, 2, \dots, N\}$ is a hyperbolic IFS with unique attractor I. Further, assume that $F_i : I \times \mathbb{R} \to \mathbb{R}$ is continuous map satisfying

$$\begin{cases} F_i(x_0, y_0) = y_{i-1}, \quad F_i(x_N, y_N) = y_i, \\ \big|F_i(x,y) - F_i(x, y^*)\big| \le s_i|y - y^*| \quad \forall\, y, y^* \in \mathbb{R},\ s_i \in (0,1). \end{cases} \tag{2.2}$$

Now define functions $W_i : I \times \mathbb{R} \to I \times \mathbb{R}$ by

$$W_i(x,y) = \big(L_i(x), F_i(x,y)\big).$$

The following is a fundamental theorem that leads to the definition of Fractal Interpolation Function.

Theorem 2.1. *([1]) The IFS $\{I \times \mathbb{R}; W_i, i = 1, 2, \dots, N\}$ has a unique attractor $G(f)$ which is the graph of a continuous function $f : I \to \mathbb{R}$ satisfying $f(x_i) = y_i$ for $i = 0, 1, \dots, N$.*

Definition 2.3. The function f that made its debut in the foregoing theorem is termed Fractal Interpolation Function (FIF) corresponding to the data $\{(x_i, y_i) : i = 0, 1, \dots, N\}$.

To obtain a functional equation for f, one may proceed as follows.
Denote by $\mathcal{C}(I)$ the space of all continuous functions defined on I endowed with the Chebyshev norm

$$\|g\|_\infty := \max\{|g(x)| : x \in I\}$$

and consider the closed (metric) subspace

$$\mathcal{C}_{y_0,y_N}(I) := \{g \in \mathcal{C}(I) : g(x_0) = y_0, g(x_N) = y_N\}.$$

Define an operator, which is a form of Read-Bajraktarivić operator, $T : \mathcal{C}_{y_0,y_N}(I) \to \mathcal{C}_{y_0,y_N}(I)$

$$(Tg)(x) = F_i\big(L_i^{-1}(x), g \circ L_i^{-1}(x)\big),\ x \in I_i,\ i \in \{1, 2, \dots, N\}.$$

Theorem 2.2. *([1]) The operator $T : \mathcal{C}_{y_0,y_N}(I) \to \mathcal{C}_{y_0,y_N}(I)$ is a contraction with a contractivity factor $s := \max\{s_i : i = 1, 2, \dots, N\}$ and the fixed point of T is the FIF f corresponding to the data $\{(x_i, y_i) : i = 0, 1, \dots, N\}$. Consequently, f satisfies the functional equation*

$$f\big(L_i(x)\big) = F_i\big(x, f(x)\big),\ x \in I,\ i = 1, 2, \dots, N.$$

The most widely studied FIFs in theory and applications are defined by IFS with maps

$$\begin{cases} L_i(x) = a_i x + b_i \\ F_i(x, y) = \alpha_i y + q_i(x), \end{cases} \tag{2.3}$$

where $|\alpha_i| < 1$ and $q_i : I \to \mathbb{R}$ is continuous function satisfying

$$q_i(x_0) = y_{i-1} - \alpha_i y_0, \quad q_i(x_N) = y_i - \alpha_i y_N.$$

The subinterval end point restraints yield

$$a_i = \frac{x_i - x_{i-1}}{x_N - x_0}, \quad b_i = \frac{x_N x_{i-1} - x_0 x_i}{x_N - x_0}.$$

The parameter α_i is called vertical scaling factor of the map W_i and the vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in (-1, 1)^N$ is refereed to as scale vector of the IFS. If q_i, $i = 1, 2, \dots, N$ are affine maps, then the FIF is termed affine FIF. In this case, $q_i(x) = q_{i0}x + q_{i1}$, where

$$q_{i0} = \frac{y_i - y_{i-1}}{x_N - x_0} - \alpha_i \frac{y_N - y_0}{x_N - x_0}, \quad q_{i1} = \frac{x_N y_{i-1} - x_0 y_i}{x_N - x_0} - \alpha_i \frac{x_N y_0 - x_0 y_N}{x_N - x_0}.$$

The following special choice of q_i in (2.3) is of special interest.

$$q_i(x) = h \circ L_i(x) - \alpha_i b(x), \tag{2.4}$$

where the height function h is a continuous interpolant to the data and base function b is a continuous function that passes though the extreme points (x_0, y_0) and (x_N, y_N). Note that in case of affine FIF, h is piecewise linear function with vertices at the data points $\{(x_i, y_i) : i = 0, 1, 2, \dots, N\}$ and b is a line joining the extremities of the interpolation interval. In contrast to traditional nonrecursive interpolants, the FIF f corresponding to (2.3) is, in general, nondifferentiable. For instance, we have

Theorem 2.3. *([14]) Let $\{(x_0, y_0), (x_1, y_1), \dots, (x_N, y_N)\}$ be an equally spaced data set in $I = [x_0, x_N] = [0, 1]$. Consider the IFS defined through the maps (2.3)-(2.4), where $h \in \mathcal{C}^1[0, 1]$, $|\alpha_i| \geq \frac{1}{N}$ for $i = 1, 2, \dots, N$, and $h'(x)$ does not agree with $y_N - y_0$ in a nonempty open subinterval of I. Then the set of points at which the corresponding FIF defined by*

$$f\big(L_i(x)\big) = h\big(L_i(x)\big) + \alpha_i(f - b)(x)$$

is not differentiable is dense in I.

However, by proper choices of elements in the IFS, a fractal function $f \in \mathcal{C}^k(I)$, $k \in \mathbb{N} \cup \{0\}$ can be constructed and this is the content of the following theorem.

Theorem 2.4. *([6]) Let $\{(x_i, y_i) : i = 0, 1, 2, \dots, N\}$ be a prescribed set of interpolation data with increasing abscissae. Consider the IFS $\{I \times \mathbb{R}; W_i, i = 1, 2, \dots, N\}$, where $W_i(x, y) = \big(L_i(x), F_i(x, y)\big)$, $L_i(x) = a_i x + b_i$, and $F_i(x, y) = \alpha_i y + q_i(x)$. Suppose that for some integer $k \geq 0$, $|\alpha_i| < a_i^k$, and $q_i \in \mathcal{C}^k(I)$ for $i = 1, 2, \dots, N$. Let*

$$F_{i,r}(x, y) = \frac{\alpha_i y + q_i^{(r)}(x)}{a_i^r};\ y_{1,r} = \frac{q_1^{(r)}(x_1)}{a_1^r - \alpha_1}, y_{N,r} = \frac{q_N^{(r)}(x_N)}{a_N^r - \alpha_N},\ r = 1, 2, \dots, k.$$

If $F_{i-1,r}(x_N, y_{N,k}) = F_{i,r}(x_0, y_{0,k})$ for $i = 2, 3, \dots, N$ and $r = 1, 2, \dots, k$, then the IFS $\{I \times \mathbb{R}; W_i : i = 1, 2, \dots, N\}$ determines a FIF $f \in \mathcal{C}^k(I)$, and $f^{(r)}$ is the fractal function determined by the IFS $\{I \times \mathbb{R}; \big(L_i(x), F_{i,r}(x, y)\big), i = 1, 2, \dots, N\}$ for $r = 1, 2, \dots, k$.

3. Fractal Functions of More General Nature

In this section we note that by simple modifications in the construction of continuous fractal interpolation function revisited in the previous section, we can break continuity and/or interpolatory property of the fractal function, providing more flexibility. Although the actual fractal function appearing in each case discussed in the sequel may be different with different properties, we shall steadfastly employ the same notation f at each appearance.

Case 1: Continuous (but not interpolatory) fractal function

For the data $\{(x_i, y_i) : i = 0, 1, 2, \dots, N\}$, consider the IFS defined by the maps

$$\begin{cases} L_i(x) = a_i x + b_i \\ F_i(x, y) = \alpha_i y + q_i(x), \end{cases} \tag{3.1}$$

where the continuous functions $q_i : I \to \mathbb{R}$ are chosen such that

$$\begin{cases} F_1(x_0, y_0) = y_0, \\ F_N(x_N, y_N) = y_N, \\ F_i(x_N, y_N) = F_{i+1}(x_0, y_0) = \tilde{y_i},\ \tilde{y_i} \in [y_i - \epsilon, y_i + \epsilon],\ i = 1, 2, \dots, N-1. \end{cases} \tag{3.2}$$

Here ϵ may be interpreted as error tolerance in measurement or noise.

Theorem 3.1. *The fractal function f corresponding to the IFS defined through (3.1)-(3.2) is continuous and satisfies $|f(x_i) - y_i| \le \epsilon$ for all $i = 0, 1, \dots, N$.*

Proof. Consider the closed metric subspace $\mathcal{C}_{y_0,y_N}(I) := \{g \in \mathcal{C}(I) : g(x_0) = y_0, g(x_N) = y_N\}$. of $\mathcal{C}(I)$. Define $T : \mathcal{C}_{y_0,y_N}(I) \to \mathcal{C}_{y_0,y_N}(I)$ by

$$(Tg)(x) = F_i\big(L_i^{-1}(x), g \circ L_i^{-1}(x)\big) = \alpha_i g\big(L_i^{-1}(x)\big) + q_i\big(L_i^{-1}(x)\big),\ x \in I_i = [x_{i-1}, x_i],\ i = 1, 2, \dots, N.$$

It follows at once that Tg is continuous on $I_i = [x_{i-1}, x_i]$ for each $i = 1, 2, \dots, N$. Bearing in mind that $L_i^{-1}(x_i) = x_N$ and $L_{i+1}^{-1}(x_i) = x_0$, for $i = 1, 2, \dots, N-1$ we have

$$\begin{aligned} (Tg)(x_i^-) &= F_i\big(L_i^{-1}(x_i), g\big(L_i^{-1}(x_i)\big) = F_i(x_N, g(x_N)) = F_i(x_N, y_N) = \tilde{y_i} \\ (Tg)(x_i^+) &= F_{i+1}\big(L_{i+1}^{-1}(x_i), g\big(L_{i+1}^{-1}(x_i)\big) = F_{i+1}(x_0, g(x_0)) = F_{i+1}(x_0, y_0) = \tilde{y_i}. \end{aligned}$$

Thus, Tg is continuous at each of the internal knots, and consequently on $I = [x_0, x_N]$. On similar lines,

$$\begin{aligned} (Tg)(x_0) &= F_1\big(L_1^{-1}(x_0), g(L_1^{-1}(x_0))\big) = F_1(x_0, g(x_0)) = F_1(x_0, y_0) = y_0 \\ (Tg)(x_N) &= F_N\big(L_N^{-1}(x_N), g(L_N^{-1}(x_N))\big) = F_N(x_N, g(x_N)) = F_N(x_N, y_N) = y_N, \end{aligned}$$

demonstrating that Tg is a well-defined map on $\mathcal{C}_{y_0,y_N}(I)$. Furthermore, for $x, y \in I_i$

$$\begin{aligned} |(Tg)(x) - (Th)(x)| &= |\alpha_i| \big| g\big(L_i^{-1}(x)\big) - h\big(L_i^{-1}(x)\big) \big| \\ &\le |\alpha_i| \|g - h\|_\infty \end{aligned}$$

Denoting $|\alpha|_\infty = \max\{|\alpha_i| : i = 1, 2, \dots, N\}$, the previous inequality stipulates

$$\|Tg - Th\|_\infty \le |\alpha|_\infty \|g - h\|_\infty,$$

proving contractivity of T in Chebyshev norm. Hence by the Banach fixed point theorem, T has a unique fixed point f. Note that $f(x_0) = y_0$, $f(x_N) = y_N$, and

$$|f(x_i) - y_i| = |(Tf)(x_i) - y_i| = \big|F_i\big(L_i^{-1}(x_i), f(L_i^{-1}(x_i))\big) - y_i\big| = |F_i(x_N, y_N) - y_i| \le \epsilon,$$

completing the proof. ☐

Remark 3.1. Treating y_0 and y_N as parameters and replacing third equation in (3.2) with the condition $F_i(x_N, y_N) = F_{i+1}(x_0, y_0)$, we obtain a continuous fractal function not attached to any data set.

Case 2: Interpolatory (but not continuous) fractal function

For a prescribed data set $\{(x_i, y_i) : i = 0, 1, \dots, N\}$, set $I = [x_0, x_N]$, $I_i = [x_{i-1}, x_i)$ for $i = 1, 2, \dots, N-1$ and $I_N = [x_{N-1}, x_N]$. Let $L_i : [x_0, x_N) \to [x_{i-1}, x_i)$ be affine maps such that $L_i(x_0) = x_{i-1}$ and $L_i(x_N^-) = x_i$ for $i = 1, 2, \dots, N-1$ and $L_N : I \to I_N$ be affine map satisfying $L_N(x_0) = x_{N-1}$ and $L_N(x_N) = x_N$. Observe that in contrast to the case of continuous fractal function, here we deal with half-open subintervals with obvious modification for the last subinterval so that each x_i belongs to a subinterval univocally. Let $q_i : I \to \mathbb{R}$ be bounded function so that the bivariate map $F_i : I \times \mathbb{R} \to \mathbb{R}$ defined by $F_i(x, y) = \alpha_i y + q_i(x)$ satisfy

$$F_i(x_0, y_0) = y_{i-1},\ i = 1, 2, \dots, N; \quad F_N(x_N, y_N) = y_N.$$

Theorem 3.2. *Consider the IFS determined by the maps L_i and F_i defined in the previous paragraph. The corresponding fractal function f is bounded and satisfies $f(x_i) = y_i$ for $i = 0, 1, \dots, N$.*

Proof. Note that the set $\mathcal{B}(I)$ of all real valued bounded functions defined on I endowed with the supremum norm is a Banach space. Consider the closed metric subspace

$$\mathcal{B}_{y_0,y_N}(I) = \{g \in \mathcal{B}(I) : g(x_0) = y_0,\ g(x_N) = y_N\}$$

of $\mathcal{B}(I)$ and define $T : \mathcal{B}_{y_0,y_N}(I) \to \mathcal{B}_{y_0,y_N}(I)$ by

$$(Tg)(x) = F_i\big(L_i^{-1}(x), g\big(L_I^{-1}(x)\big)\big) = \alpha_i g\big(L_i^{-1}(x)\big) + q_i\big(L_i^{-1}(x)\big),\ x \in I_i,\ i = 1, 2, \dots, N.$$

It is plain to see that Tg is a bounded function. For $i = 1, 2, \dots, N$, x_{i-1} belongs univocally to I_i and using definition of Tg

$$(Tg)(x_{i-1}) = F_i\big(L_i^{-1}(x_{i-1}), g\big(L_i^{-1}(x_{i-1})\big)\big) = F_i\big(x_0, g(x_0)\big) = F_i\big(x_0, y_0\big) = y_{i-1}.$$

Further, x_N belongs to I_N and using N-th piece of the definition of Tg one obtains

$$(Tg)(x_N) = F_N\big(L_N^{-1}(x_N), g\big(L_N^{-1}(x_N)\big)\big) = F_N\big(x_N, g(x_N)\big) = F_N\big(x_N, y_N\big) = y_N.$$

Therefore, Tg is well-defined and maps into $\mathcal{B}_{y_0,y_N}(I)$. Following the proof of previous theorem we assert that T is a contraction, and consequently the Banach fixed point theorem ensures the existence of a unique fixed point f. It follows at once that $f(x_i) = (Tf)(x_i) = y_i$ for $i = 0, 1, \dots, N$, delivering the promised result. ☐

Remark 3.2. Due to the lack of "join-up" condition $F_i(x_N, y_N) = F_{i+1}(x_0, y_0)$, $i = 1, 2, \dots, N-1$, the fractal function f in the preceding theorem has a jump discontinuity at each of the internal knots (and hence possibly at many other points). Same is the case, even if q_i, $i = 1, 2, \dots, N$ are assumed to be continuous on I. Later, to derive additional properties of bounded (discontinuous) fractal function, we shall assume that the maps q_i, $i = 1, 2, \dots, N$ involved in the IFS are Lipschitz continuous.

Case 3: Discontinuous fractal function

Here we consider fractal function in $\mathcal{B}(I)$, which is not attached to a data set. Let $\{x_0, x_1, \dots, x_N\}$ be a partition of $I = [x_0, x_N]$ satisfying $x_0 < x_1 < \dots < x_N$. As in the previous case, let $I_i = [x_{i-1}, x_i)$ for $i = 1, 2, \dots, N-1$ and $I_N = [x_{N-1}, x_N]$. Let $L_i : [x_0, x_N) \to [x_{i-1}, x_i)$ be affinities such that $L_i(x_0) = x_{i-1}$ and $L_i(x_N-) = x_i$ for $i = 1, 2, \dots, N-1$ and $L_N : I \to I_N$ be affine map satisfying $L_N(x_0) = x_{N-1}$ and $L_N(x_N) = x_N$. For $i = 1, 2, \dots, N$, let q_i be bounded function. Note that we do not require any additional conditions such as join-up conditions and end point conditions for the bivariate maps $F_i(x, y) = \alpha_i y + q_i(x)$.

Theorem 3.3. *The fractal function f corresponding to the IFS defined via the maps L_i and F_i of the previous paragraph is bounded and satisfies the self-referential equation*

$$f\big(L_i(x)\big) = \alpha_i f(x) + q_i(x).$$

Proof. Define a map $T : \mathcal{B}(I) \to \mathcal{B}(I)$ by

$$(Tg)(x) = F_i\big(L_i^{-1}(x), g\big(L_i^{-1}(x)\big)\big) = \alpha_i g\big(L_i^{-1}(x)\big) + q_i\big(L_i^{-1}(x)\big),\ x \in I_i,\ i = 1, 2, \dots, N.$$

Since g and q_i are in the linear space $\mathcal{B}(I)$, it follows readily that $Tg \in \mathcal{B}(I)$ and T is well-defined. As noted previously, for $g, h \in \mathcal{B}(I)$

$$\begin{aligned}\big|(Tg)(x) - (Th)(x)\big| &= |\alpha_i|\big|g\big(L_i^{-1}(x)\big) - h\big(L_i^{-1}(x)\big)\big| \\ &\le |\alpha_i|\|g - h\|_\infty \\ &\le |\alpha|_\infty\|g - h\|_\infty,\end{aligned}$$

and hence

$$\|Tg - Th\|_\infty \le |\alpha|_\infty\|g - h\|_\infty.$$

Therefore, T is a contraction and its fixed point f enjoys the self-referential equation

$$f\big(L_i(x)\big) = \alpha_i f(x) + q_i(x),$$

completing the proof. □

Remark 3.3. It is worth to note that instead of constant scaling factors, we may employ scaling functions $\alpha_i : I \to \mathbb{R}$ with suitable conditions to provide fractal functions with more flexibility. To to the least, we may assume that each α_i is bounded and $\|\alpha\|_\infty := \max\{\|\alpha_i\|_\infty : i = 1, 2, \dots, N\} < 1$

Remark 3.4. Since fractal function f appearing in Theorem 3.3 is bounded, it belongs to $\mathcal{L}^\infty(I)$ and also to $\mathcal{L}^p(I)$ for $1 \le p < \infty$. In particular, f is Lebesgue integrable.

Remark 3.5. As pointed out in reference [1] for a continuous function, given $f \in \mathcal{B}(I)$ one may choose $b \in \mathcal{B}(I)$ and consider $q_i(x) := f(L_i(x)) - \alpha_i b(x)$. The corresponding fractal function, which is termed α-fractal function, denoted by f^α provides the self-referential analogue of f. In case b depends linearly on f, then the correspondence $f \mapsto f^\alpha$ provides a linear operator on $\mathcal{B}(I)$, extending the notion of α-fractal operator (see [18]) to the space $\mathcal{B}(I)$.

Remark 3.6. On lines similar to Theorem 3.1, for a prescribed data set $\{(x_i, y_i) : i = 0, 1, \dots, N\}$ by proper choices of α_i and q_i, we can construct a fractal function $f \in \mathcal{B}(I)$ (not necessarily continuous) that approximates the data in the sense that $|f(x_i) - y_i| \le \epsilon$ for $i = 0, 1, \dots, N$. For instance, Lipschitz map q_i can be taken as affinities $q_i(x) = q_{i0}x + q_{i1}$, $i = 1, 2, \dots, N$ and the coefficients thereof can be selected so that the function values at the knots

$$\begin{aligned}&f(x_0) = \frac{q_{10}x_0 + q_{11}}{1 - \alpha_1}, \quad f(x_N) = \frac{q_{N0}x_N + q_{N1}}{1 - \alpha_N}, \\ &f(x_i) = \alpha_{i+1} f(x_0) + q_{i+i,0}x_0 + q_{i+1,1}, \quad i = 1, 2, \dots, N - 1\end{aligned}$$

are close enough to y_i, $i = 0, 1, \dots, N$.

Example 3.1. We now illustrate previous results by constructing examples of continuous and discontinuous fractal functions which interpolate or approximate the set of data $\{(0, 0), (0.5, 0.5), (1, 0)\}$. Fig. 1 shows the affine fractal function in the standard setting (i.e., continuous and interpolatory) with scale vector $\alpha = (0.75, 0.75)$. Note that the graph of the FIF has Minkowski dimension $D \approx 1.585$ obtained as the unique solution of (see [1])

$$\sum_{i=1}^{N} |\alpha_i| a_i^{D-1} = 1.$$

Fig. 2 represents continuous affine fractal approximants, where the scale vector is taken to be $\alpha = (0.5, 0.5)$ and coefficients appearing in the affinities $q_i(x) = q_{i0}x + q_{i1}$ are chosen such that $|f(x_i) - y_i| < \epsilon$ with $\epsilon = 0.1, 0.05$, and 0.005. Fig. 3 displays discontinuous fractal interpolation function corresponding to the data set wherein scaling vector is $\alpha = (0.5, 0.5)$ and the coefficient c_1 appearing in the affine map is chosen (at random) as $\frac{1}{8}$. In Fig. 4, we break both continuity and interpolatory conditions inherent in a traditional affine FIF. The coefficients of affinities are randomly chosen so that the graph passes close to the given data, that is, we construct discontinuous fractal function f satisfying $|f(x_i) - y_i| < 0.1$.

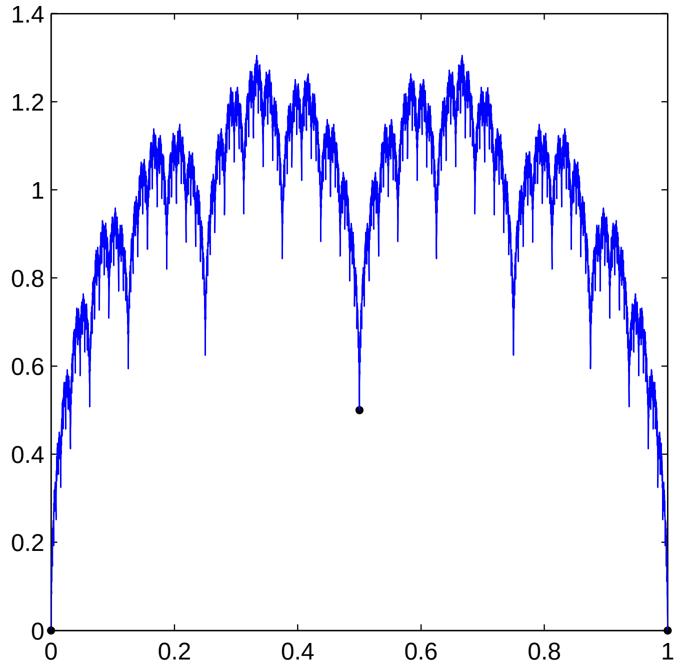

FIGURE 1. Graph of a continuous affine fractal interpolation function.

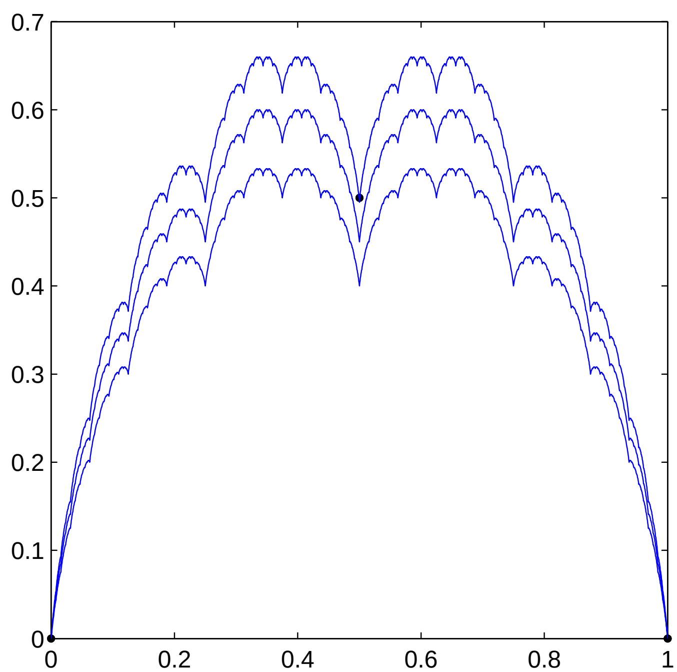

FIGURE 2. Graphs of continuous affine fractal (approximation) functions.

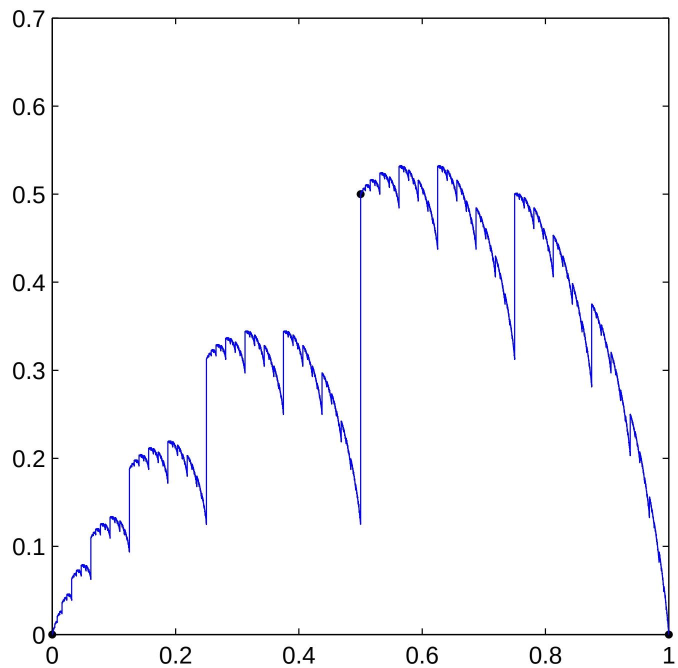

FIGURE 3. Graph of a discontinuous affine fractal interpolation function (The vertical lines display discontinuities).

4. Some Properties of Discontinuous Fractal Functions

Recall that a continuous fractal interpolation function can be obtained either as a fixed point of operators defined on suitable function spaces or as attractors of IFSs. In the previous section, we obtained discontinuous fractal functions as fixed points of RB-operators. Here we first prove that the graph $G(f)$ of this bounded fractal function f is related to the attractor of the IFS $\{I \times \mathbb{R}; W_i, i = 1, 2, \dots, N\}$. Further, we prove that the set of points of discontinuities of f has Lebesgue measure zero. Our analysis follows closely ([7], Theorem 3.5, p. 403) which proves a similar result for local fractal functions. However, we note that for the validity of the theorem in [7] suitable additional condition, for instance, Lipschitz continuity, is to be imposed on the maps λ_i (which takes the role of q_i in the present setting) that seems to be missing.
As a prelude, let us recall a pair of definitions.
If $f : X \to \mathbb{R}$ is a function on a metric space (X, d), then the oscillation of f on an open set U is

$$\omega_f(U) = \sup_{x \in U} f(x) - \inf_{x \in U} f(x) = \sup_{a,b \in U} |f(a) - f(b)|$$

and the oscillation of f at a point x^* is defined as

$$\omega_f(x^*) = \lim_{\epsilon \to 0} \omega_f\big(B_\epsilon(x^*)\big),$$

where $B_\epsilon(x^*)$ is the open ball at x^* with radius ϵ defined by $B_\epsilon(x^*) = \{x \in X : d(x, x^*) < \epsilon\}$. Note that f is continuous at x^* if and only if $\omega_f(x^*) = 0$.

Theorem 4.1. *Consider $W_i : I \times \mathbb{R} \to I \times \mathbb{R}$ defined by $W_i(x, y) = \big(L_i(x), F_i(x, y)\big)$ with L_i and F_i as in Theorem 3.3. Further assume that q_i is Lipschitz continuous with Lipschitz constant Q_i and $Q = \max\{Q_i : i = 1, 2, \dots, N\}$. Then the IFS $\{I \times \mathbb{R}; W_i, i = 1, 2, \dots, N\}$ is contractive with respect to a metric d_θ on $\mathbb{R}^2$ defined by*

$$d_\theta\big((x, y), (x', y')\big) = |x - x'| + \theta|y - y'|,$$

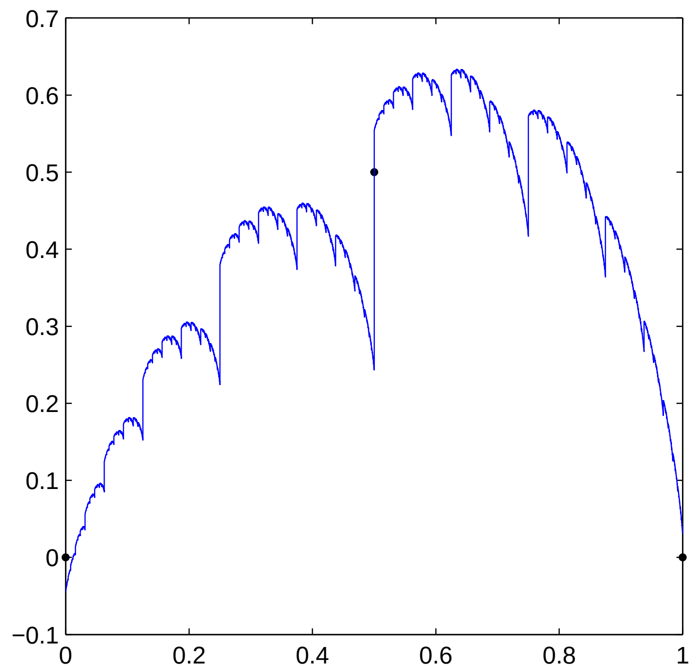

FIGURE 4. Graph of a discontinuous affine fractal (approximation) function (The vertical lines display discontinuities).

where $0 < \theta < \dfrac{1 - \max_{1\le i\le N} a_i}{Q}$. *Furthermore, the unique attractor of this IFS is the closure of the graph of* f.

Proof. Let $(x,y),(x',y') \in I \times \mathbb{R}$. We have

$$\begin{aligned}
d_\theta\big(W_i(x,y), W_i(x',y')\big) &= d_\theta\Big(\big(L_i(x), \alpha_i y + q_i(x)\big), \big(L_i(x'), \alpha_i y' + q_i(x')\big)\Big)\\
&= |L_i(x) - L_i(x')| + \theta\big|\alpha_i y + q_i(x) - \big(\alpha_i y' + q_i(x')\big)\big|\\
&\le a_i|x - x'| + \theta\big[|\alpha_i||y - y'| + |q_i(x) - q_i(x')|\big]\\
&= (a_i + \theta Q_i)|x - x'| + \theta|\alpha_i||y - y'|\\
&\le \max\{a_i + \theta Q_i, |\alpha_i|\}\big[|x - x'| + \theta|y - y'|\big].
\end{aligned}$$

For θ as mentioned in the statement of the theorem, it follows that $K := \max\{a_i + \theta Q_i, |\alpha_i|\} < 1$ and consequently that W_i, $i = 1,2,\ldots,N$ are contraction maps. That is, the IFS $\{I \times \mathbb{R}; W_i, i = 1,2,\ldots,N\}$ is contractive, and hence it has a unique attractor, say A. Let $G(f) := \{(x, f(x)) : x \in I\}$ denote the graph of f and $\mathcal{H}(I \times \mathbb{R})$ denote the space of all nonempty compact subsets of $I \times \mathbb{R}$ endowed with Hausdorff metric. Consider the Hutchinson operator $W : \mathcal{H}(I \times \mathbb{R}) \to \mathcal{H}(I \times \mathbb{R})$ defined by

$$W(B) = \cup_{i=1}^N W_i(B)$$

We have to show that $W(\overline{G(f)}) = \overline{G(f)}$. Let us extend W to $2^{I\times\mathbb{R}}$, the powerset of $I \times \mathbb{R}$. With a slight abuse of notation we shall denote the extension also by W. Consider $(x,y) \in \overline{G(f)}$. Then there exists a sequence of points $x_n \in I$ such that $\lim_{n\to\infty}(x_n, f(x_n)) = (x,y)$. Since $I = \cup_{i=1}^N I_i$ and $I_i \cap I_j = \emptyset$ for all $i \ne j$, for each fixed x_n, we have $x_n = L_i(x_n')$, and hence

$$\begin{aligned}
(x_n, f(x_n)) = \big(L_i(x_n'), f(L_i(x_n'))\big) &= \big(L_i(x_n'), F_i(x_n', f(x_n'))\big)\\
&= W_i\big(x_n', f(x_n')\big) \in W(G(f)) \subseteq W(\overline{G(f)}).
\end{aligned}$$

Consequently, $(x,y) \in \overline{W(\overline{G(f)})} = W(\overline{G(f)})$, showing that

$$\overline{G(f)} \subseteq W(\overline{G(f)}).$$

Next to show that $W(\overline{G(f)}) \subseteq \overline{G(f)}$, we first show that $W(\tilde{G}) \subseteq G(f)$, where $\tilde{G} := G(f) \setminus \big\{(x_i, f(x_i)), i = 0, 1, \dots, N\big\}$. Let $(x,y) \in W(\tilde{G})$. Then $(x,y) \in W_i(\tilde{G})$ for some $i \in \{1, 2, \dots, N\}$. Therefore, there is an (x', y') such that $x' \in I \setminus \{x_0, x_1, \dots, x_N\}$, $y' = f(x')$, $x = L_i(x')$, and

$$y = F_i(x', y') = \alpha_i f(x') + q_i(x') = \alpha_i f\big(L_i^{-1}(x)\big) + q_i\big(L_i^{-1}(x)\big) = f(x),$$

so that $(x,y) \in G(f)$, proving $W(\tilde{G}) \subseteq G(f)$. Using this and continuity of the Hutchinson map

$$W(\overline{G(f)}) = W(\overline{\tilde{G}}) \subseteq \overline{W(\tilde{G})} \subseteq \overline{G(f)},$$

completing the proof. □

The foregoing theorem shows that the graph of a bounded discontinuous fractal interpolant f can be approximated by the chaos game algorithm [4].

Remark 4.1. Note that in the case of continuous fractal function f, the graph $G(f)$ is closed, and hence the attractor coincides with $G(f)$.

This next theorem which is just a slight variant of the Collage theorem (see [2]) hints towards the choices of maps W_i so that the bounded fractal function f is close to a prescribed $\Phi \in \mathcal{B}(I)$.

Theorem 4.2. *Let the mappings W_i, $i = 1, 2, \dots, N$ in the IFS used to generate the bounded discontinuous fractal function f (Cf. Theorem 3.3) be chosen such that $\|\Phi - T\Phi\|_\infty < \epsilon$, where T is the RB-operator whose fixed point yields f. Then*

$$\|\Phi - f\|_\infty < \frac{\epsilon}{1 - |\alpha|_\infty}.$$

Proof. By the Banach fixed point theorem, the fractal function $f = \lim_{m\to\infty} T^m g$, where $g \in \mathcal{B}(I)$ is arbitrary. Therefore we see that

$$\begin{aligned}
\|\Phi - f\|_\infty = \|\Phi - \lim_{m\to\infty} T^m\Phi\|_\infty &= \lim_{m\to\infty} \|\Phi - T^m\Phi\|_\infty \\
&= \lim_{m\to\infty} \Big\| \sum_{i=1}^{m} (T^{i-1}\Phi - T^i\Phi) \Big\|_\infty \\
&\le \lim_{m\to\infty} \sum_{i=1}^{m} \|T^{i-1}\Phi - T^i\Phi\|_\infty \\
&= \lim_{m\to\infty} \sum_{i=1}^{m} \|T^{i-1}(\Phi - T\Phi)\|_\infty \\
&\le \lim_{m\to\infty} \sum_{i=1}^{m} |\alpha|_\infty^{i-1} \|\Phi - T\Phi\|_\infty \\
&< \frac{\epsilon}{1 - |\alpha|_\infty},
\end{aligned}$$

providing the assertion. □

Theorem 4.3. *The set of points of discontinuities for the function f (Cf. Theorem 3.3) is a Lebesgue null set. In particular, f is Riemann integrable.*

Proof. Since f is obtained by the application of Banach fixed point theorem on the Read-Bajraktarević operator T, $f = \lim_{n\to\infty} g_n$, where $g_n = T g_{n-1}$, and $g_0 \in \mathcal{B}(I)$ is arbitrary. We choose $g_0 \equiv 1$, the constant function on $I = [x_0, x_N]$. Note that

$$g_n\big(L_i(x)\big) = \alpha_i g_{n-1}(x) + q_i(x),\ x \in I,\ i = 1, 2, \dots, N.$$

The function g_1 may have finite jump discontinuities at the internal knot points $x_1, x_2, \dots, x_{N-1}$. In general, g_n may have finite jump discontinuities at the internal knot points $x_1, x_2, \dots, x_{N-1}$ of the partition and possibly at the image points $L_{i_1} \circ L_{i_2} \circ \dots L_{i_{n-1}}(x_r)$ of the internal knots. Therefore by a simple arithmetic, it follows that the n-th pre-fractal function g_n has at most $(N-1)(N^{n-1}+1)$ points of discontinuity in I. Let A_n denote the set of points of discontinuities

of g_n, then $|A_n| \leq (N-1)(N^{n-1}+1)$. Further, assume $A = \cup_{n=1}^{\infty} A_n$, so that $|A| \leq \aleph_0$, that is A is countable. We prove that f is continuous at $x \in I \setminus A$. Choose an $\epsilon > 0$. Taking into account that q_i, $i = 1, 2, \dots, N$ are Lipschitz functions with Lipschitz constants Q_i and $Q = \max\{Q_i : i = 1, 2, \dots, N\}$, from the fixed point equation

$$f\big(L_i(x)\big) = \alpha_i f(x) + q_i(x)$$

it follows that

$$\begin{aligned} \omega_f\big(L_i(I)\big) &= \sup_{x,y \in I} \big|f(L_i(x)) - f(L_i(y))\big| \\ &= \sup_{x,y \in I} \big|\alpha_i(f(x) - f(y)) + q_i(x) - q_i(y)\big| \qquad (4.1) \\ &\leq |\alpha|_\infty \omega_f(I) + Q|I|, \end{aligned}$$

where $|\alpha|_\infty := \max\{|\alpha_i| : i = 1, 2, \dots, N\}$ and $|I|$ is the length of I.

Let $\Omega = \{1, 2, \dots, N\}$ and let Ω^∞ be the set of all infinite sequences $\sigma = \sigma_1\sigma_2\sigma_3\dots$ of elements in Ω. Then (Ω, d_Ω), refereed to as code space, is a compact metric space, with the metric d_Ω defined by $d_\Omega(\sigma, \omega) = 2^{-k}$, where k is the least index for which $\sigma_k \neq \omega_k$. For any finite code $\sigma|_k = \sigma_1\sigma_2\dots\sigma_k$ from (4.1) we obtain

$$\begin{aligned} \omega_f\big(L_{\sigma|_k}(I)\big) &= \omega_f\big(L_{\sigma_1} \circ L_{\sigma_2} \circ \cdots \circ L_{\sigma_k}(I)\big) \\ &\leq |\alpha|_\infty w_f\big(L_{\sigma_2} \circ \cdots \circ L_{\sigma_k}(I)\big) + Qa^{k-1}|I|, \end{aligned}$$

which on recursion yields

$$\omega_f\big(L_{\sigma|_k}(I)\big) = |\alpha|_\infty^k \omega_f(I) + Q|I| \frac{a^k}{\big||\alpha|_\infty - a\big|}. \qquad (4.2)$$

Consider the contractive IFS $\{I; L_i, i = 1, 2, \dots, N\}$ with attractor I. The limit $\lim_{k\to\infty} L_{\sigma_1} \circ L_{\sigma_2} \circ \dots L_{\sigma_k}(a)$ is a single point independent of $a \in I$ and the coding map $\pi : \Omega^\infty \to I$

$$\pi(\sigma) := \lim_{k\to\infty} L_{\sigma_1} \circ L_{\sigma_2} \circ \dots L_{\sigma_k}(a)$$

is continuous and surjective. Therefore there exists $\sigma \in \Omega^\infty$ such that

$$x = \pi(\sigma) = \cap_{k=1}^{\infty} L_{\sigma|_k}(I).$$

For any $k \in N$, there exists a compact interval I_k such that

$$x \in I_k \subset \cap_{i=1}^{k} L_{\sigma|_i}(I).$$

Set $J := L_{\sigma|_k}^{-1}(I_k)$, where $\sigma|_k^{-1} = L_{\sigma_k}^{-1} \circ L_{\sigma_{k-1}}^{-1} \circ \cdots \circ L_{\sigma_1}^{-1}$. In view of (4.2) we obtain

$$\omega_f(I_k) = \omega_f\big(L_{\sigma|_k}(J)\big) \leq |\alpha|^k \omega_f(I) + Q|J| \frac{a^k}{\big||a| - |\alpha|_\infty\big|}.$$

Since f is bounded on I, $|J| \leq x_N - x_0$, and $a^k \to 0$ as $k \to \infty$, we can choose k to be large enough so that each summand in the previous inequality is less than $\frac{\epsilon}{2}$. Consequently, $\omega_f(I_k) < \epsilon$. Since ϵ is arbitrary, we deduce that $\omega_f(x) = 0$, and hence f is continuous at $x \in I \setminus A$. That f is Riemann integrable follows now from a standard result in analysis which states that a real-valued bounded function f is Riemann integrable if and only if set of points of discontinuities for f has Lebesgue measure zero, see for instance, [22]. □

Theorem 4.4. *Let f be the discontinuous fractal function given in Theorem 3.3. For each $m \in \mathbb{N} \cup \{0\}$, the moment integral*

$$f_m = \int_I x^m f(x)\ \mathrm{d}x$$

can be explicitly evaluated recursively in terms of the lower moment integrals $f_{m-1}, f_{m-2}, \dots, f_0$, the scaling factors α_i, and the moment

$$Q_m = \int_I x^m Q(x)\ \mathrm{d}x,$$

where the function $Q : I \to \mathbb{R}$ is defined as

$$Q(x) = q_i \circ L_i^{-1}(x), \ \textit{for } x \in I_i.$$

Proof. In view of the previous theorem it follows at once that the moment integrals are well-defined in the Riemann sense. With a series of self-explanatory steps we have

$$\begin{aligned}
f_m &= \sum_{i=1}^{N} \int_{I_i} x^m f(x)\ \mathrm{d}x \\
&= \sum_{i=1}^{N} \int_{I_i} x^m \big[\alpha_i f\big(L^{-1}(x)\big) + q_i\big(L_i^{-1}(x)\big)\big]\ \mathrm{d}x \\
&= \sum_{i=1}^{N} a_i \alpha_i \int_I (a_i \tilde{x} + b_i)^m f(\tilde{x})\ \mathrm{d}\tilde{x} + \int_I x^m Q(x)\ \mathrm{d}x \\
&= \sum_{i=1}^{N} \sum_{k=0}^{m} \alpha_i a_i^{k+1} b_i^{m-k} \binom{m}{k} f_k + Q_m,
\end{aligned}$$

which may be recast as

$$f_m = \frac{\sum_{k=0}^{m-1} \binom{m}{k} f_k \sum_{i=1}^{N} \alpha_i a_i^{k+1} b_i^{m-k} + Q_m}{1 - \sum_{i=1}^{N} \alpha_i a_i^{m+1}}.$$

Also, in particular

$$f_0 = \int_I f(x)\ \mathrm{d}x = \frac{\int_I Q(x)\ \mathrm{d}x}{1 - \sum_{i=1}^{N} a_i \alpha_i}.$$

Since $a_i = \dfrac{x_i - x_{i-1}}{x_N - x_0}$, we get $\sum_{i=1}^{N} a_i = 1$ and hence on account of $|\alpha_i| < 1$ it follows that $\sum_{i=1}^{N} a_i \alpha_i < 1$. □

We can extend the discontinuous fractal function f supported on I to whole $\mathbb{R}$ by defining the extension to zero off I, which will also be denoted by f. Now the functional equations satisfied by integral transforms of these discontinuous fractal functions can be easily obtained, which may be of interest for variety of reasons, see also [1]. The general integral transform of a "well behaved" function f is defined as

$$\hat{f}(s) = \int_{\mathbb{R}} K(x,s) f(x)\ \mathrm{d}x,$$

where $K(x,s)$ is a suitable function, referred to as kernel of the transformation. Using the functional equation for fractal function f one obtains

$$\begin{aligned}
\hat{f}(s) &= \int_I K(x,s) f(x)\ \mathrm{d}x \\
&= \sum_{i=1}^{N} \int_{I_i} K(x,s) \big[\alpha_i f\big(L_i^{-1}(x)\big) + q_i\big(L_i^{-1}(x)\big]\ \mathrm{d}x \\
&= \sum_{i=1}^{N} a_i \alpha_i \int_I K\big(L_i(x), s\big) f(x)\ \mathrm{d}x + \hat{Q}(s),
\end{aligned}$$

where $\hat{Q}(s)$ is the integral transform of the function $Q : I \to \mathbb{R}$ defined by $Q(x) = q_i\big(L_i^{-1}(x)\big)$ for $x \in I_i$.

This being said, it is tempting to examine transform of a fractal function with some special choices of kernel functions.

Case 1: Laplace transform

Here $K(x,s) = e^{-sx}$ for $x > 0$. Then

$$\begin{aligned}
\hat{f}(s) &= \sum_{i=1}^{N} a_i \alpha_i \int_I e^{-s(a_i x + b_i)} f(x)\ \mathrm{d}x + \hat{Q}(s) \\
&= \sum_{i=1}^{N} a_i \alpha_i e^{-s b_i} \hat{f}(a_i s) + \hat{Q}(s).
\end{aligned}$$

Case 2: Stieltjes transform
Taking $K(x,s)=\frac{1}{s-x}$ we obtain

$$\begin{aligned}\hat{f}(s) &= \sum_{i=1}^{N} a_i\alpha_i \int_I \frac{1}{s-(a_ix+b_i)} f(x)\ \mathrm{d}x + \hat{Q}(s)\\ &= \sum_{i=1}^{N} \alpha_i \hat{f}\Big(\frac{1}{a_i}(s-b_i)\Big) + \hat{Q}(s).\end{aligned}$$

Case 3: Fourier transform
For the kernel $K(x,s)=e^{jsx}$, where j is the square root of -1, we have

$$\begin{aligned}\hat{f}(s) &= \sum_{i=1}^{N} a_i\alpha_i \int_I e^{js(a_ix+b_i)} f(x)\ \mathrm{d}x + \hat{Q}(s)\\ &= \sum_{i=1}^{N} a_i\alpha_i e^{jsb_i} \hat{f}(a_is) + \hat{Q}(s).\end{aligned}$$

In case of uniformly spaced knot sequence we obtain an explicit expression as follows. Denoting $\Lambda(s)=\frac{1}{N}\sum_{i=1}^{N}\alpha_i e^{jsb_i}$, for equidistant knots, i.e., for $a_i=\frac{1}{N}$ previous equation yields

$$\hat{f}(s) = \Lambda(s)\hat{f}(\frac{s}{N}) + \hat{Q}(s). \tag{4.3}$$

Applying Equation (4.3) recursively, we obtain

$$\hat{f}(s) = \Big[\prod_{i=1}^{k}\Lambda(\frac{s}{N^{i-1}})\Big]\hat{f}(\frac{s}{N^k}) + \sum_{i=0}^{k-1}\Big[\prod_{m=1}^{i}\Lambda(\frac{s}{N^{m-1}})\Big]\hat{Q}(\frac{s}{N^i}), \tag{4.4}$$

where the empty product $\prod_{m=1}^{0}\Lambda\big(\frac{s}{N^{m-1}}\big)=1$. Note that the previous equation extemporizes [Equation 3.4, [17], p. 177].
We have

$$|\Lambda(s)| \le \frac{1}{N}\sum_{i=1}^{N}|\alpha_i e^{jsb_i}| \le |\alpha|_\infty,$$

and therefore

$$\Big|\prod_{i=1}^{k}\Lambda(\frac{s}{N^{i-1}})\Big| \le |\alpha|_\infty^k \to 0 \text{ as } k\to 0.$$

The previous observation in conjunction with boundedness of $\hat{f}$ asserts that the first summand in (4.4) approaches zero as $k\to\infty$. Since

$$\Big|\prod_{m=1}^{i}\Lambda(\frac{s}{N^{m-1}})\hat{Q}(\frac{s}{N^m})\Big| \le M|\alpha|_\infty^i,$$

where M is such that $|\hat{Q}(s)|\le M$ for all $s\in\mathbb{R}$, from (4.4) we obtain

$$\hat{f}(s) = \sum_{i=0}^{\infty}\Big[\prod_{m=1}^{i}\Lambda(\frac{s}{N^{m-1}})\Big]\hat{Q}(\frac{s}{N^i}).$$

5. Fractal Histopolation

Suppose that a sequence of strictly increasing knots $\{x_0,x_1,\dots,x_N\}$ and a histogram $F=\{f_1,f_2,\dots,f_N\}$, where $f_i\in\mathbb{R}$ is the frequency for the class $[x_{i-1},x_i)$, $i=1,2,\dots,N$ are given. For $i=1,2,\dots,N$, let $h_i:=x_i-x_{i-1}$ represent the step size. In fractal histopolation, we match average pixel intensities with our fractal function, in contrast to matching point value as done with interpolation. That is, we seek for an integrable fractal function f satisfying "area" matching condition

$$\int_{x_{i-1}}^{x_i} f(x)\ \mathrm{d}x = h_if_i. \tag{5.1}$$

Consider the IFS defined by the maps

$$L_i(x) = a_i x + b_i, \; F_i(x,y) = \alpha_i y + q_i(x), \quad i = 1, 2, \dots, N, \tag{5.2}$$

where $|\alpha_i| < 1$ and $q_i : I \to \mathbb{R}$ is Lipschitz continuous map. From Theorem 3.3 and Theorem 4.3 it follows that the corresponding fractal function is Riemann integrable and satisfies

$$f(x) = \alpha_i f\big(L_i^{-1}(x)\big) + q_i^{-1}\big(L_i^{-1}(x)\big), \quad x \in I_i, \; i = 1, 2, \dots, N.$$

The parameters that can be varied are scaling factors α_i and functions q_i, $i = 1, 2, \dots, N$. The histopolation condition prescribed in Equation (5.1) necessitates

$$\begin{aligned} h_i f_i &= \int_{I_i} f(x) \, \mathrm{d}x \\ &= \int_{I_i} \Big[\alpha_i f\big(L_i^{-1}(x)\big) + q_i\big(L_i^{-1}(x)\big)\Big] \, \mathrm{d}x \\ &= \alpha_i a_i \int_I f(x) \, \mathrm{d}x + a_i \int_I q_i(x) \, \mathrm{d}x \\ &= \alpha_i a_i \sum_{i=1}^N h_i f_i + a_i \int_I q_i(x) \, \mathrm{d}x, \; i = 1, 2, \dots, N. \end{aligned} \tag{5.3}$$

Assume $\sum_{i=1}^N h_i f_i \neq 0$. If q_i are a priori fixed maps, then in the previous equation only unknown is the scaling factor α_i, which is obtained via

$$\alpha_i = \frac{h_i f_i - a_i \int_I q_i(x) \, \mathrm{d}x}{a_i \sum_{i=1}^N h_i f_i}, \quad i = 1, 2, \dots, N.$$

However, this solution may not be feasible, since we require that $|\alpha_i| < 1$ for all $i = 1, 2, \dots, N$. (or a less stringent condition $\big[\sum_{i=1}^N a_i |\alpha_i|^p\big]^{\frac{1}{p}}$ if we work in $\mathcal{L}^1(I)$ instead of $\mathcal{B}(I)$). Thus, in principle, the problem demands a constrained optimization. In practice, for a quicker solution, we can fix scaling factors α_i a priori and treat q_i as unknown functions to be determined suitably so that the corresponding bounded integrable fractal function f satisfies Equation (5.1).

Proposition 5.1. *Let a sequence of strictly increasing knots $\{x_0, x_1, \dots, x_N\}$ and a histogram $F = \{f_1, f_2, \dots, f_N\}$ be given. Consider the IFS $\{I \times \mathbb{R}; W_i : i = 1, 2, \dots, N\}$ defined through the maps given in Equation (5.2). Assume that the scaling factors are selected at random so that $|\alpha_i| < 1$ for $i = 1, 2, \dots, N$. The corresponding fractal function solves the histopolation problem $\big($Cf. Equation (5.1)$\big)$ if and only if the function q_i satisfies*

$$\int_I q_i(x) \, \mathrm{d}x = \frac{h_i f_i - \alpha_i a_i \sum_{i=1}^N h_i f_i}{a_i}, \quad i = 1, 2, \dots, N. \tag{5.4}$$

Proof. Necessary condition follows at once from Equation (5.3). From the functional equation for f one obtains (see Theorem 4.4)

$$\int_I f(x) \, \mathrm{d}x = \frac{\sum_{i=1}^N a_i \int_I q_i(x) \, \mathrm{d}x}{1 - \sum_{i=1}^N a_i \alpha_i}.$$

Therefore,

$$\begin{aligned} \int_{I_i} f(x) \, \mathrm{d}x &= \alpha_i a_i \int_I f(x) \, \mathrm{d}x + a_i \int_I q_i(x) \, \mathrm{d}x \\ &= \alpha_i a_i \frac{\sum_{i=1}^N a_i \int_I q_i(x) \, \mathrm{d}x}{1 - \sum_{i=1}^N a_i \alpha_i} + a_i \int_I q_i(x) \, \mathrm{d}x. \end{aligned}$$

Substituting the stated condition on q_i in the previous equation we can deduce that $\int_{I_i} f(x) \, \mathrm{d}x = h_i f_i$, completing the proof. □

In what follows, we shall outline some choices for q_i satisfying condition in Proposition 5.1. For instance, taking q_i as affine maps $q_i(x) = q_{i0}x + q_{i1}$, $i = 1, 2, \dots, N$, we obtain

$$h_i f_i = \alpha_i a_i \sum_{i=1}^{N} h_i f_i + a_i(x_N - x_0)\Big[\frac{q_{i0}}{2}(x_N + x_0) + q_{i1}\Big], \quad i = 1, 2, \dots, N. \tag{5.5}$$

Since $\alpha_i \in (-1, 1)$ are chosen as parameters as in the case of affine fractal interpolation function, the above system consists of N linear equations in $2N$ unknowns q_{i0} and q_{i1}, $i = 1, 2, \dots, N$. Treating q_{i0} also as parameters we obtain

$$q_{i1} = \frac{h_i f_i - a_i \alpha_i \sum_{i=1}^{N} h_i f_i}{a_i(x_N - x_0)} - \frac{q_{i0}}{2}(x_N + x_0), \quad i = 1, 2, \dots, N. \tag{5.6}$$

Next suppose that we are interested to construct a continuous fractal histopolant corresponding to strictly increasing knots $\{x_0, x_1, \dots, x_N\}$ and a histogram $F = \{f_1, f_2, \dots, f_N\}$. Choose y_0 and y_N arbitrary. Recall from Section 3 that the fractal function corresponding to Equation (5.2) is continuous if map q_i, $i = 1, 2, \dots, N$ satisfies

$$\begin{cases} q_1(x_0) = y_0(1 - \alpha_1) \\ q_N(x_N) = y_N(1 - \alpha_N) \\ \alpha_{i+1} y_0 + q_{i+1}(x_0) = \alpha_i y_N + q_i(x_N), \quad i = 1, 2, 3, \dots, N-1. \end{cases} \tag{5.7}$$

Then the next proposition follows at once.

Proposition 5.2. *Let a sequence of strictly increasing knots $\{x_0, x_1, \dots, x_N\}$ and a histogram $F = \{f_1, f_2, \dots, f_N\}$ be given. Consider the IFS $\{I \times \mathbb{R}; W_i : i = 1, 2, \dots, N\}$ defined through the maps given in Equation (5.2). Assume that the scaling factors are selected at random so that $|\alpha_i| < 1$ for $i = 1, 2, \dots, N$. The corresponding fractal function is continuous and solves the histopolation problem (Cf. Equation (5.1)) if the function q_i satisfies system of equations governed by Equation(5.4) and Equation(5.7).*

It naturally raised the question of solvability of the system mentioned in the foregoing Proposition. To this end, we make some remarks. If we take $q_i(x) = q_{i0}x + q_{i1}$ and $\alpha_i \in (-1, 1)$ as adjustable parameters, then Proposition 5.2 provides a system of $2N + 1$ linear equations with $2N + 2$ unknowns y_0, y_N, q_{i0}, and q_{i1}, $i = 1, 2, \dots, N$, choosing one of the unknowns, say y_0, arbitrarily, the resulting square system of linear equations may be solved. Alternatively, one may proceed as follows. Assume values to $q_1(x_N), q_2(x_N), \dots, q_{N-1}(x_N)$, y_0, and y_N so that Equation (5.7) specifies $q_i(x_0)$ and $q_i(x_N)$ for $i = 1, 2, \dots, N$. Let $q_i(x_0) = \beta_i$, and $q_i(x_N) = \gamma_i$. Equation 5.4 is equivalent to

$$\int_I q_i(x)\ \mathrm{d}x = (x_N - x_0)\Big[\frac{h_i f_i - \alpha_i a_i \sum_{i=1}^{N} h_i f_i}{a_i(x_N - x_0)}\Big],\ i = 1, 2, \dots, N.$$

Therefore, in this case, the problem reduces to that of solving N histopolation problems with boundary conditions

$$\begin{cases} \displaystyle\int_I q_i(x)\ \mathrm{d}x = (x_N - x_0)\Big[\frac{h_i f_i - \alpha_i a_i \sum_{i=1}^{N} h_i f_i}{a_i(x_N - x_0)}\Big] \\ q_i(x_0) = \beta_i,\ q_i(x_N) = \gamma_i, \quad i = 1, 2, \dots, N, \end{cases} \tag{5.8}$$

for which one can employ methods of histopolation by traditional nonrecursive functions, see, for instance, [12]. Note that the corresponding fractal histopolant f is continuous, but non-differentiable in general. For a special choice of q_i used in the definition of α-fractal function (see Remark 3.5), construction of continuous fractal histopolant f seems to be rather easy. To this end, let $g \in \mathcal{C}(I)$ and $b \in \mathcal{C}(I)$ be such that $b \not\equiv g$, $b(x_0) = g(x_0)$ and $b(x_N) = g(x_N)$. Consider the IFS $\{I \times \mathbb{R}; W_i(x, y) = (L_i(x), F_i(x, y)), i = 1, 2, \dots, N\}$, where

$$F_i(x, y) = \alpha_i y + g \circ L_i(x) - \alpha_i b(x).$$

Corresponding fractal function $f \in \mathcal{C}(I)$ satisfies

$$f(x) = g(x) + \alpha_i (f - b)\big(L^{-1}(x)\big).$$

Histopolation condition in Equation (5.4) reads as

$$\int_{I_i} g(x)\ \mathrm{d}x - \alpha_i \int_I b(x)\ \mathrm{d}x = a_i^{-1}\Big[h_i f_i - a_i\alpha_i \sum_{i=1}^{N} h_i f_i\Big].$$

As mentioned earlier, one can solve $N+1$ histopolation problems $\int_{I_i} g(x) = \frac{h_i f_i}{a_i}$ and $\int_I b(x) = \sum_{i=1}^{N} h_i f_i$ with boundary conditions $b(x_0) = g(x_0)$ and $b(x_N) = g(x_N)$. It is worthwhile to mention that a fractal histospline f of continuity $\mathcal{C}^k$ for the knot sequence $\{x_0, x_1, \dots, x_N\}$ can be obtained by differentiating a $\mathcal{C}^{k+1}$-continuous fractal spline interpolating the data $\{(x_i, y_i) : i = 0, 1, \dots, N\}$, where, for instance, $y_0 = 0$ and $y_i = y_{i-1} + h_i f_i$ for $i = 1, 2, \dots, N$. Fractal splines interpolating a prescribed data have received much attention in the literature, see for example [6, 10, 20].

Example 5.1. Consider the knot sequence $\{0, 1/2, 1\}$ and a histogram $F = \{2, 3\}$. We construct area true approximants of the histogram F by using integrable fractal functions. Consider the IFS defined by the maps

$$L_1(x) = \frac{1}{2}x,\ L_2(x) = \frac{1}{2}x + \frac{1}{2},\quad F_1(x,y) = \frac{1}{2}y + x + \frac{1}{4},\ F_2(x,y) = \frac{1}{2}y + 2x + \frac{3}{4}.$$

Here the coefficients q_{10} and q_{20} appearing in the affinities are taken (at random) as 1 and 2 respectively, and the other coefficients are calculated using Equation (5.6). Resulting discontinuous fractal function given by

$$f(x) = \begin{cases} \frac{1}{2}f(2x) + 2x + \frac{1}{4} & \text{if} \quad x \in [0, \frac{1}{2}), \\ \\ \frac{1}{2}f(2x-1) + 4x - \frac{5}{4} & \text{if} \quad x \in [\frac{1}{2}, 1], \end{cases}$$

satisfies histopolation conditions $\int_0^{\frac{1}{2}} f(x)\ \mathrm{d}x = 1$ and $\int_{\frac{1}{2}}^1 f(x)\ \mathrm{d}x = \frac{3}{2}$, see Figure 5. Assume that the problem demands a continuous fractal histopolant. Bearing Proposition 5.2 in mind, taking scale factors $\alpha_1 = \alpha_2 = 0.5$ and assuming the value y_0 of the histopolant at the end point $x_0 = 0$ to be 0, we solve the linear system to obtain $q_{10} = \frac{3}{2}$, $q_{11} = 0$, $q_{20} = -\frac{3}{2}$ and $q_{21} = \frac{5}{2}$. Corresponding continuous fractal histopolant satisfying the functional equation

$$f(x) = \begin{cases} \frac{1}{2}f(2x) + 3x & \text{if} \quad x \in [0, \frac{1}{2}), \\ \\ \frac{1}{2}f(2x-1) - 3x + 4 & \text{if} \quad x \in [\frac{1}{2}, 1], \end{cases}$$

is depicted in Figure 6. The $\mathcal{C}^1$-continuous histospline in Figure 7 is obtained by differentiating a cubic spline fractal function g interpolating the data set $\{(0,0), (\frac{1}{2}, 1), (1, \frac{5}{2})\}$. For details on cubic spline FIF, the reader may consult [6, 10]. Corresponding smooth histopolant satisfies the self-referential equation

$$f(x) = \begin{cases} 0.4f(2x) - 1.4616x^2 + 0.4x + 2.0436 & \text{if} \quad x \in [0, \frac{1}{2}), \\ \\ 0.4f(2x-1) - 0.3384x^2 + 1.676x + 0.9404 & \text{if} \quad x \in [\frac{1}{2}, 1]. \end{cases}$$

Fractal histopolants may be used to model planar data with prescribed Minkowski dimension which controls the selection of scaling factors. Minkowski and Hausdorff dimensions of a more general fractal function, for instance, bounded discontinuous fractal function, continue to remain as an open problem. There are many other strategies for identification of free parameters in fractal histopolation, and quite often the particular nature of the modeling problem states the type of optimization to be employed. These also deserve future investigation.

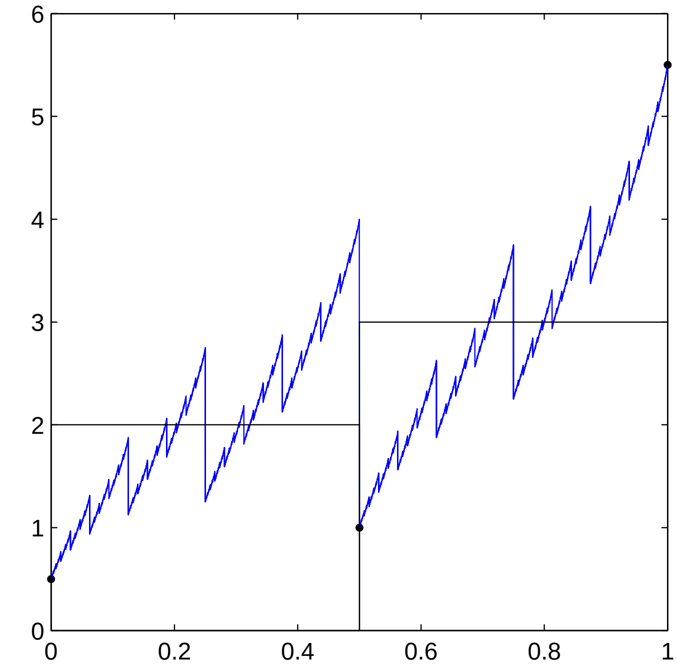

FIGURE 5. Histogram and discontinuous affine fractal histopolant.

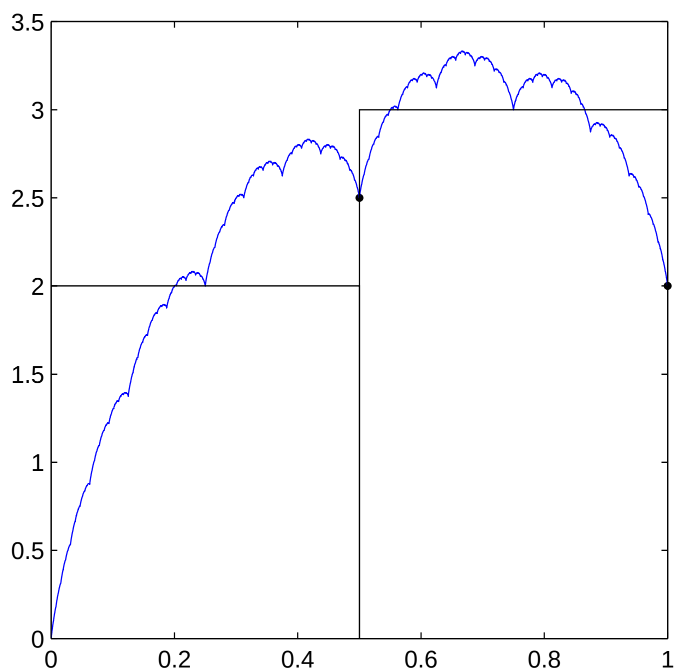

FIGURE 6. Histogram and continuous affine fractal histopolant.

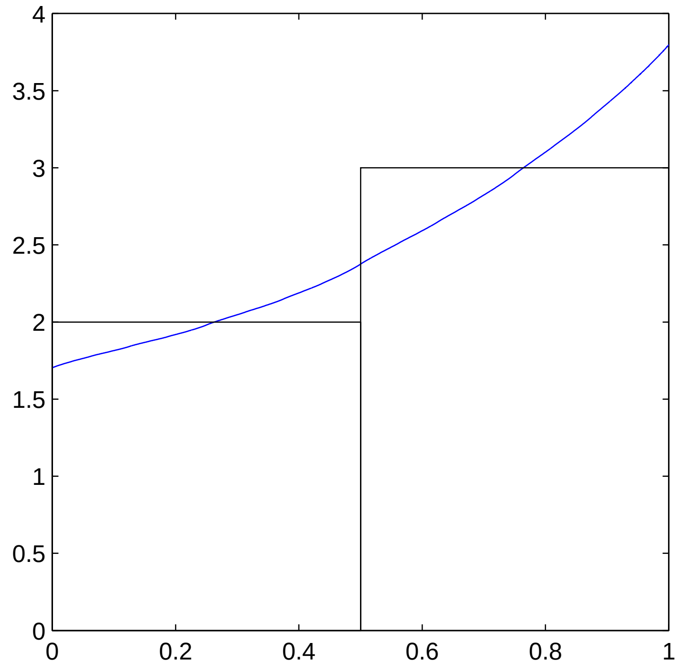

FIGURE 7. Histogram and $\mathcal{C}^1$-continuous histospline.

References

[1] M.F. Barnsley, Fractal functions and interpolation, *Constr. Approx.* **2** (1986) 303–329.

[2] M.F. Barnsley, Fractals Everywhere, Academic Press, Inc., 1988.

[3] M.F. Barnsley, B. Harding, A. Vince and P. Viswanathan, arXive 1412.3871V2, March 2015.

[4] M. F. Barnsley and A. Vince, The chaos game on a general iterated function system, *Ergod. Th. & Dynam. Syst.* **31** (2011) 1073–1079.

[5] M.F. Barnsley, J. Elton, D. Hardin and P.R. Massopust, Hidden variable fractal interpolation functions, *SIAM J. Math. Anal.* **20 (5)** (1989) 1218–1242.

[6] M.F. Barnsley and A.N. Harrington, The calculus of fractal interpolation functions, *J. Approx. Theory* **57(1)** (1989) 14–34.

[7] M.F. Barnsley, M. Hegland and P. Massopust, Numerics and Fractals, *Bull. Inst. Math. Acad. Sinica* **9(3)** (2014) 389–430.

[8] S. Basu, E. Foufoula-Georgiou and F. Porté-Agel, Synthetic turbulence, fractal interpolation, and large-eddy simulation. *Phys. Rev. E* **70** (2004) 026310.

[9] P. Bouboulis and L. Dalla, A general construction of fractal interpolation functions on grids of R^n, *European J. Appl. Math.* **18** (2007) 449–476.

[10] A.K.B. Chand and P. Viswanathan, A constructive approach to cubic Hermite Fractal Interpolation Function and its constrained aspects, *BIT Numer. Math.* **53** (2013) 841–865.

[11] O.I. Craciunescu, S.K. Das, J.M. Poulson and T.V. Samulski, Three-dimensional tumor perfusion reconstruction using fractal interpolation functions, *IEEE Trans. Biomed. Eng.* **48(4)** (2001) 462–73.

[12] M. Fischer, P. Oja and H. Trossmann, Comonotone shape-preserving spline histopolation, *J. Comp. Appl. Math.* **200** (2007) 127–139.

[13] J.S. Geronimo and D. Hardin, Fractal interpolation surfaces and a related 2-D multiresolution analysis, *J. Math. Anal. Appl.* **176** (1993) 561–586.

[14] J. Li and W. Su, The smoothness of fractal interpolation functions on $\mathbb{R}$ and on p-series local fields, *Discrete Dyn. Nat. Soc.* **2014**, Article ID 904576, 10 pages, (2014).

[15] B.B. Madelbrot, The fractal geometry of nature, Freeman, San Francisco. 1982.

[16] P. Massopust, Fractal functions, fractal surfaces, and wavelets, Academic Press, Inc., San Diego, CA, 1994.

[17] P. Massopust, Fractal functions and applications, *Chaos, Solitons & Fractals*, **8(2)** (1997) 171–190.

[18] M.A. Navascués, Fractal polynomial interpolation, *Z. Anal. Anwend.*, **25(2)** (2005) 401–418.

[19] M.A. Navascués, Fractal functions of discontinous approximation, *J. Basic Appl. Sci.*, **10** (2014) 173–176.

[20] M.A. Navascués and M.V. Sebastián, Smooth fractal interpolation, *J. Inequal. Appl.* **Article ID 78734** (2006) 1–20.

[21] J.R. Price and M.H. Hayes, Resampling and reconstruction with fractal interpolation functions, *IEEE Signal Processing Letters*, **5(9)** (1998) 228–230.
[22] W. Rudin, Real and complex analysis, McGraw-Hill Company Limited, New York, 1966.
[23] M.V. Sebastián and M.A. Navascués, relation between fractal dimension and Fourier transform - electroencephalographic study using spectral and fractal parameters, *Int. J. Comp. Math.*, **85(3-4)** (2008) 657–665.
[24] J.L. Véhel, K. Daoudi and E. Lutton, Fractal modeling of speech signals, Fractals **2(3)** (1994) 379–382.
[25] P. Viswanathan and A.K.B. Chand, Fractal rational functions and their approximation properties, *J. Approx. Theory* **185**, (2014) 31–50.
[26] H.Y. Wang and J.S. Yu, Fractal interpolation functions with variable parameters and their analytical properties, *J. Approx. Theory* **175** (2013) 1–18.
[27] L. Yong and T. Xin, Fractal fitting research on stock prices. In: 2008 Congress on image and signal processing, **4** 49–53.

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